

b -ARY EXPANSIONS OF ALGEBRAIC NUMBERS

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ABSTRACT. In this paper we give a generalization of the main results in [1, 2] about b -ary expansions of algebraic numbers. As a byproduct we get a large class of new transcendence criteria. One of our corollaries implies that b -ary expansions of linearly independent irrational algebraic numbers are quite independent. Motivated by this result, we propose a generalized Borel conjecture.

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1. INTRODUCTION

Let $b \geq 2$ be a fixed integer. Any real number ω has a unique b -ary expansion:

$$\omega = [\omega] + \sum_{i \geq 1} a_i b^{-i} = [\omega] + 0.a_1 a_2 \cdots,$$

where $a_i \in \{0, 1, \dots, b-1\}$ and the set $\{i \mid a_i \neq b-1\}$ is infinite. ω is called *normal* to base b if, for every $k \geq 1$, every block of k digits from $\{0, 1, \dots, b-1\}$ occurs in the b -ary expansion of ω with frequency $1/b^k$.

A classical theorem of Borel [3] says that almost all real numbers are normal to base b . In [4] Borel made the conjecture that all irrational algebraic numbers are normal to base b . It seems that this conjecture is far from the reach of modern mathematics. Let $p(\omega, n)$ be the number of distinct blocks

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of length n occurring in the b -ary expansion of ω . It follows from Borel's conjecture that $p(\omega, n) = b^n$ for any irrational algebraic numbers ω . But even this corollary is too difficult. A large breakthrough in this direction is due to Adamczewski and Bugeaud [1]. Before introducing their results, we need some preparations.

We say that an infinite word $\mathbf{a} = a_1a_2\cdots$ of elements from $\{0, 1, \dots, b-1\}$ has *long repetition* if the following condition is satisfied, where the length of a finite word A is denoted by $|A|$.

Condition 1.1. *There exist three sequences of finite nonempty words $\{A_n\}_{n \geq 1}$, $\{A'_n\}_{n \geq 1}$, $\{B_n\}_{n \geq 1}$ such that:*

- (i) *for any $n \geq 1$, $A_nB_nA'_nB_n$ is a prefix of \mathbf{a} ;*
- (ii) *the sequence $\{|B_n|\}_{n \geq 1}$ is strictly increasing;*
- (iii) *there exists a positive constant L such that*

$$(|A_n| + |A'_n|)/|B_n| \leq L,$$

for every $n \geq 1$.

One of the main results in [1] is the following:

Theorem 1.2. *The b -ary expansion of an irrational algebraic number has no long repetitions.*

It follows directly from this theorem that

$$\lim_{n \rightarrow +\infty} \frac{p(\omega, n)}{n} = +\infty,$$

where ω is an irrational algebraic number. This result, though far from the conjecture that $p(\omega, n) = b^n$, is indeed a great advance comparing with the previous result [6] that

$$\lim_{n \rightarrow +\infty} p(\omega, n) - n = +\infty.$$

In [2], Adamczewski and Bugeaud further explored the independence of b -ary expansions of two irrational algebraic numbers α and β .

Let

$$\mathbf{a} = a_1a_2a_3\cdots$$

and

$$\mathbf{a}' = a'_1a'_2a'_3\cdots$$

be two infinite words of elements from $\{0, 1, \dots, b-1\}$. The following is a condition about the pair $(\mathbf{a}, \mathbf{a}')$:

Condition 1.3. *There exist three sequences of finite nonempty words $\{A_n\}_{n \geq 1}$, $\{A'_n\}_{n \geq 1}$, $\{B_n\}_{n \geq 1}$ such that:*

- (i) *for any $n \geq 1$, the word A_nB_n is a prefix of \mathbf{a} and the word A'_nB_n is a prefix of \mathbf{a}' ;*
- (ii) *the sequence $\{|B_n|\}_{n \geq 1}$ is strictly increasing;*

(iii) *there exists a positive constant L such that*

$$(|A_n| + |A'_n|)/|B_n| \leq L,$$

for every $n \geq 1$.

The main result in [2] is:

Theorem 1.4. *Let α and α' be two irrational algebraic numbers. If their b -ary expansions*

$$\alpha = [\alpha] + 0.a_1a_2\cdots,$$

and

$$\alpha' = [\alpha'] + 0.a'_1a'_2\cdots$$

satisfy Condition 1.3, then the two infinite words

$$\mathbf{a} = a_1a_2a_3\cdots$$

and

$$\mathbf{a}' = a'_1a'_2a'_3\cdots$$

have the same tail.

In this paper, we show that for a fix irrational algebraic number α , and a fix nontrivial linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0,$$

where $a_1, a_2, \dots, a_n \in \mathbb{Z}$, the b -ary expansion of α can not have n disjoint long sub-words which are correlated by the above linear equation. When the linear equation is $x_1 - x_2 = 0$, we recover Theorem 1.2. Similar result holds for several algebraic numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over \mathbb{Q} . Applying this result to a pair of algebraic numbers quickly implies Theorem 1.4. Our method is different from that of [1, 2]. In particular, we do not use rational approximations to algebraic numbers. Instead, we deduce all things from a single theorem about greatest common divisor of a big sum and a pow of b (Theorem 2.1).

This paper is structured as follows: In Section 2, we state the main results of this paper after some preparations. In Section 3, we supply all the proofs. In Sections 4, we propose a generalized Borel conjecture and some other questions.

2. MAIN RESULTS

Throughout this paper, let $b \geq 2$ be a fixed integer. All the irrational algebraic numbers we consider lie in the interval $(0, 1)$. All the words considered in this paper mean words of elements from $\{0, 1, \dots, b-1\}$. For an irrational algebraic number α , we always identify the b -ary expansions

$$\alpha = 0.a_1a_2a_3\cdots$$

with the infinite word

$$\mathbf{a} = a_1a_2a_3\cdots.$$

The length of a finite word A is denoted by $|A|$. For any real number a , $[a]$ and $\{a\}$ denote respectively the integer part and the fractional part of a . Finally, for two integers a and b , denote the greatest common divisor of a and b by $G.C.D(a, b)$.

Before giving the main result, we introduce some new definitions and notations.

Given n positive integers a_1, a_2, \dots, a_n , an array of **nonzero** rational numbers $\{b_{i,j}\}$, where $1 \leq i \leq n$, $1 \leq j \leq a_i$, will be called an (a_1, a_2, \dots, a_n) -array. For each (a_1, a_2, \dots, a_n) -array $\{b_{i,j}\}$, Set

$$[\{b_{i,j}\}] = \min_{i,j} (b_{i,j}),$$

$$\langle \{b_{i,j}\} \rangle = \min_{\substack{1 \leq i \leq n \\ 1 \leq j \leq a_i - 1}} (b_{i,j+1} - b_{i,j}),$$

and set

$$\overline{\langle \{b_{i,j}\} \rangle} = \min_{(i,j) \neq (i',j')} (|b_{i,j} - b_{i',j'}|).$$

Let L be a positive real number. An (a_1, a_2, \dots, a_n) -array $\{b_{i,j}\}$ is *L-admissible* if the following conditions are satisfied :

- (i) $b_{i,j}$ is a positive integer for $1 \leq i \leq n$, $1 \leq j \leq a_i$;
- (ii) $b_{i,j} < b_{i,j+1}$ for $1 \leq i \leq n$, $1 \leq j \leq a_i - 1$;
- (iii) $\max_{i,j} (b_{i,j}) \leq L \min_{i,j} (b_{i,j})$.

Now we are in the position to state a theorem from which all the results of this paper can be derived:

Theorem 2.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n irrational algebraic numbers and let ϵ, L , and M be fix positive numbers. Let P be an real valued function defined on the set of all L -admissible $(\underbrace{1, \dots, 1}_n)$ -arrays such that $|P(D)| < M$ for every D . For each L -admissible $(\underbrace{1, \dots, 1}_n)$ -array $D = \{d_{i,1}\}$, set*

$$f(D) = \sum_{i=1}^n \alpha_i b^{d_{i,1}}.$$

Then if both $[D]$ and $\overline{\langle D \rangle}$ are sufficiently large, we have

$$G.C.D([f(D) + P(D)], b^{[D]}) < b^{\epsilon [D]}.$$

A direct consequence of Theorem 2.1 is the following:

Theorem 2.2. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n algebraic numbers, such that*

$$1, \alpha_1, \alpha_2, \dots, \alpha_n$$

are linearly independent over \mathbb{Q} . Let a_1, a_2, \dots, a_n be n positive integers, let $C = \{c_{i,j}\}$ be a fixed (a_1, a_2, \dots, a_n) -array, and let ϵ, L and M be fixed positive numbers. Let P be an real valued function defined on the set of all

L -admissible (a_1, a_2, \dots, a_n) -arrays such that $|P(D)| < M$ for every D .
For each L -admissible (a_1, a_2, \dots, a_n) -array $D = \{d_{i,j}\}$, set

$$f(D) = \sum_{i=1}^n \sum_{j=1}^{a_i} \alpha_i c_{i,j} b^{d_{i,j}}.$$

Then when both $\lfloor D \rfloor$ and $\langle D \rangle$ are sufficiently large, we have

$$G.C.D(\lfloor f(D) + P(D) \rfloor, b^{\lfloor D \rfloor}) < b^{\epsilon \lfloor D \rfloor}.$$

The following two special cases of Theorem 2.2 is more convenient for applications.

Theorem 2.3. Let α be an irrational algebraic number. Let n a positive integer. Let $C = \{c_i\}$ be a fixed (n) -array, and let ϵ and L be fixed positive numbers. For each L -admissible (n) -array $D = \{d_i\}$, set

$$f(D) = \sum_{i=1}^n c_i [\alpha b^{d_i}].$$

Then when both $\lfloor D \rfloor$ and $\langle D \rangle$ are sufficiently large, we have

$$G.C.D(\lfloor f(D) \rfloor, b^{\lfloor D \rfloor}) < b^{\epsilon \lfloor D \rfloor}.$$

Theorem 2.4. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n algebraic numbers, such that $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over \mathbb{Q} . Let ϵ and L be fixed positive numbers. For each L -admissible $\underbrace{(1, \dots, 1)}_n$ -array $D = \{d_i\}$, set

$$f(D) = \sum_{i=1}^n [\alpha_i b^{d_i}].$$

Then when $\lfloor D \rfloor$ is sufficiently large, we have

$$G.C.D(f(D), b^{\lfloor D \rfloor}) < b^{\epsilon \lfloor D \rfloor}.$$

We need some preliminaries before giving more specific corollaries of Theorem 2.2.

Let a_1, a_2, \dots, a_m be m fixed non-zero integers. Let \mathbf{a} be an infinite word of elements from $\{0, 1, \dots, b-1\}$. For any finite non-empty word $s_m \cdots s_1 s_0$, set

$$\overline{s_m \cdots s_1 s_0} = \sum_{i=0}^m s_i b^i.$$

The following is a condition about \mathbf{a} and the numbers a_1, a_2, \dots, a_m :

Condition 2.5. There exist $2m$ sequences of finite nonempty words

$$\{A_i^1\}_{i \geq 1}, \dots, \{A_i^m\}_{i \geq 1}, \{B_i^1\}_{i \geq 1}, \dots, \{B_i^m\}_{i \geq 1}$$

such that:

- (i) for each $i \geq 1, j \leq m$, the word $A_i^j B_i^j$ is a prefix of the word \mathbf{a} ;

(ii) for each $i \geq 1$,

$$|B_i^1| = \cdots = |B_i^m| = k_i,$$

and the sequence $\{k_i\}_{i \geq 1}$ is strictly increasing;

(iii) for each $i \geq 1$,

$$|A_i^1| < \cdots < |A_i^m|,$$

and for each $j < m$ the sequence $\{|A_i^{j+1}| - |A_i^j|\}_{i \geq 1}$ is strictly increasing;

(iv) for each $i \geq 1$,

$$\sum_{j=1}^m a_j \overline{B_i^j} \equiv 0 \pmod{b^{k_i}};$$

(v) there exists a positive constant L such that

$$|A_i^j|/|B_i^j| \leq L$$

for $i \geq 1, j \leq m$.

Let $\mathbf{a}^1, \dots, \mathbf{a}^m$ be m infinite words of elements from $\{0, 1, \dots, b-1\}$. The second condition is about $\mathbf{a}^1, \dots, \mathbf{a}^m$ and the numbers a_1, a_2, \dots, a_m :

Condition 2.6. *There exist $2m$ sequences of finite nonempty words*

$$\{A_i^1\}_{i \geq 1}, \dots, \{A_i^m\}_{i \geq 1}, \{B_i^1\}_{i \geq 1}, \dots, \{B_i^m\}_{i \geq 1}$$

such that:

(i) for each $i \geq 1, j \leq m$, the word $A_i^j B_i^j$ is a prefix of the word \mathbf{a}^j ;

(ii) for each $i \geq 1$,

$$|B_i^1| = \cdots = |B_i^m| = k_i,$$

and the sequence $\{k_i\}_{i \geq 1}$ is strictly increasing;

(iii) for each $i \geq 1$,

$$\sum_{j=1}^m a_j \overline{B_i^j} \equiv 0 \pmod{b^{k_i}};$$

(iv) there exists a positive constant L such that

$$|A_i^j|/|B_i^j| \leq L$$

for $i \geq 1, j \leq m$.

Theorem 2.7. *Let α be an irrational algebraic number, and let a_1, a_2, \dots, a_m be m fixed non-zero integers. Then the b -ary expansion of α does not satisfy Condition 2.5.*

When $m = 2$, $a_1 = -a_2 = 1$, the congruence

$$\overline{B_i^1} \equiv \overline{B_i^2} \pmod{b^{k_i}}$$

in Condition 2.5 forces $B_i^1 = B_i^2$. In this way we recover Theorem 1.2 immediately.

Theorem 2.7 and its variants immediately yield a large class of new transcendence criteria. The following are two simplest examples:

Let Φ be the set of finite words of length ≥ 2 on the alphabet $\{0, 1, \dots, b-1\}$. Let s be a nonzero integer. For any $\mathbf{a} \in \Phi$ of length k , let \mathbf{a}^1 and \mathbf{a}^2 be the two sub-words of \mathbf{a} such that $\mathbf{a}^1\mathbf{a}^2$ is the prefix of \mathbf{a} and

$$|\mathbf{a}^1| = |\mathbf{a}^2| = \lfloor k/2 \rfloor.$$

Let \mathbf{a}^3 be the unique word of length $\lfloor k/2 \rfloor$ in Φ satisfying

$$\overline{\mathbf{a}^1} + \overline{\mathbf{a}^2} \equiv \overline{\mathbf{a}^3} \pmod{b^{\lfloor k/2 \rfloor}},$$

and let \mathbf{a}^4 be the unique word of length k in Φ satisfying

$$s\overline{\mathbf{a}} \equiv \overline{\mathbf{a}^4} \pmod{b^k}.$$

We define two operations f and g on Φ by

$$f(\mathbf{a}) = \mathbf{a}\mathbf{a}^3$$

and

$$g(\mathbf{a}) = \mathbf{a}\mathbf{a}^4.$$

Let $F(\mathbf{a})$ be the limit of $f^n(\mathbf{a})$, and let $G(\mathbf{a})$ be the limit of $g^n(\mathbf{a})$ for $n \rightarrow +\infty$. Now Theorem 2.7 immediately implies the following two transcendence criteria.

Theorem 2.8. *For any $\mathbf{a} \in \Phi$, set*

$$F(\mathbf{a}) = a_1a_2 \dots$$

and

$$G(\mathbf{a}) = a'_1a'_2 \dots$$

Then neither

$$\sum_{i=0}^{\infty} a_i b^{-i} = a_1 b^{-1} + a_2 b^{-2} + \dots$$

nor

$$\sum_{i=0}^{\infty} a'_i b^{-i} = a'_1 b^{-1} + a'_2 b^{-2} + \dots$$

can be irrational algebraic number.

Remark 2.9. *In fact, both*

$$\sum_{i=0}^{\infty} a_i b^{-i} = a_1 b^{-1} + a_2 b^{-2} + \dots$$

and

$$\sum_{i=0}^{\infty} a'_i b^{-i} = a'_1 b^{-1} + a'_2 b^{-2} + \dots$$

in the above theorem are transcendental except for some extreme cases.

Now we consider simultaneous b -ary expansions of several algebraic numbers.

Theorem 2.10. *Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be m algebraic numbers, such that*

$$1, \alpha_1, \alpha_2, \dots, \alpha_m$$

are linearly independent over \mathbb{Q} . Let a_1, a_2, \dots, a_m be m fixed non-zero integers. Then Condition 2.6 is not satisfied by the b -ary expansions of $\alpha_1, \alpha_2, \dots, \alpha_m$.

When $m = 2$, $a_1 = -a_2 = 1$, Condition 2.6 reduces to Condition 1.3.

It can be derived directly from Theorem 2.2 that Theorems 2.7 and 2.10 have a common generalization of mixed type. It is a little tedious to write down the full result. A simplest case says that:

Theorem 2.11. *Let a_1, a_2, a_3 be three fixed non-zero integers. Let α and α' be two algebraic numbers, such that $1, \alpha, \alpha'$ are linearly independent over \mathbb{Q} . Then their b -ary expansions*

$$\alpha = 0.a_1a_2 \cdots,$$

and

$$\alpha' = 0.a'_1a'_2 \cdots$$

can not satisfy Condition 2.12

Condition 2.12. *There exist six sequences of finite nonempty words*

$$\{A_i^1\}_{i \geq 1}, \{A_i^2\}_{i \geq 1}, \{A_i^3\}_{i \geq 1}, \{B_i^1\}_{i \geq 1}, \{B_i^2\}_{i \geq 1}, \{B_i^3\}_{i \geq 1}$$

such that:

- (i) *for each $i \geq 1$, both $A_i^1B_i^1$ and $A_i^2B_i^2$ are prefixes of $a_1a_2 \cdots$;*
- (ii) *for each $i \geq 1$, $A_i^3B_i^3$ is a prefix of $a'_1a'_2 \cdots$;*
- (iii) *for each $i \geq 1$,*

$$|B_i^1| = |B_i^2| = |B_i^3| = k_i,$$

and the sequence $\{k_i\}_{i \geq 1}$ is strictly increasing;

- (iv) *for each $i \geq 1$,*

$$\sum_{j=1}^3 a_j \overline{B_i^j} \equiv 0 \pmod{b^{k_i}};$$

- (v) *for each $i \geq 1$, $|A_i^1| < |A_i^2|$, and the sequence $\{|A_i^2| - |A_i^1|\}_{i \geq 1}$ is strictly increasing;*
- (vi) *there exists a positive constant L such that*

$$|A_i^j|/|B_i^j| \leq L$$

for $i \geq 1, j \leq 3$.

3. PROOFS OF THE MAIN RESULTS

As in [1, 2], our proofs are dependent upon the following p -adic version of the Schmidt subspace theorem [8, 9]. Let $|\cdot|_p$ is the p -adic absolute value on \mathbb{Q} , normalized by $|p|_p = p^{-1}$. we pick an extension of $|\cdot|_p$ to $\overline{\mathbb{Q}}$.

Theorem 3.1. *Let $n \geq 1$ be an integer, and let S be a finite set of places on \mathbb{Q} containing the infinite place. For every*

$$\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1},$$

set

$$\|\mathbf{x}\| = \max_i \{|x_i|\}.$$

For every $p \in S$, let $L_{0,p}(\mathbf{x}), \dots, L_{n,p}(\mathbf{x})$ be linearly independent linear forms in $n+1$ variables with algebraic coefficients. Then for any positive number ϵ , the solutions $\mathbf{x} \in \mathbb{Z}^{n+1}$ of the inequality

$$\prod_{p \in S} \prod_{i=1}^{n+1} |L_{i,p}(\mathbf{x})|_p \leq \|\mathbf{x}\|^{-\epsilon}$$

lie in finitely many proper linear subspaces of \mathbb{Q}^{n+1}

Proof of Theorem 2.1. For the infinite place ∞ and for each prime $p \mid b$, we will introduce $n+1$ linearly independent linear forms as follows:

For every

$$\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1},$$

set

$$L_{0,\infty}(\mathbf{x}) = -x_0 + \sum_{i=1}^n \alpha_i x_i,$$

and for each $1 \leq i \leq n$, set

$$L_{i,\infty}(\mathbf{x}) = x_i.$$

For each prime $p \mid b$ and each $0 \leq i \leq n$, set

$$L_{i,p}(\mathbf{x}) = x_i.$$

Assume that there exists an infinite sequence $\{D_k\}_{k \geq 1} = \{\{d_{i,1,k}\}_i\}_{k \geq 1}$ of L -admissible $\underbrace{(1, \dots, 1)}_n$ -arrays, such that both $\lfloor D_k \rfloor$ and $\langle \overline{D_k} \rangle$ tend to infinity, and

$$R_k \geq b^{\epsilon \lfloor D_k \rfloor},$$

where

$$Q_k = [f(D_k) + P(D_k)],$$

and

$$R_k = G.C.D(Q_k, b^{\lfloor D_k \rfloor}).$$

Set

$$\mathbf{X}_k = \frac{1}{R_k} (Q_k, b^{d_{1,1,k}}, \dots, b^{d_{n,1,k}}) \in \mathbb{Z}^{n+1}.$$

Then we have

$$|L_{0,\infty}(\mathbf{X}_k)| = \left| \frac{1}{R_k} (f(D_k) - [f(D_k) + P(D_k)]) \right| < \left| \frac{2(M+1)}{R_k} \right|.$$

Direct estimation shows that

$$\|\mathbf{X}_k\| < b^{2\max_i(d_{i,1,k})} \leq b^{2L\lfloor D_k \rfloor},$$

for k sufficiently large. By product formula (cf.[7, p.99]),

$$|L_{i,\infty}(\mathbf{X}_k)| \prod_{p|b} |L_{i,p}(\mathbf{X}_k)|_p = \left| \frac{b^{d_{i,1,k}}}{R_k} \right| \prod_{p|b} \left| \frac{b^{d_{i,1,k}}}{R_k} \right|_p = 1,$$

for $i = 1, \dots, n$. Hence

$$\prod_{\substack{p=\infty \\ \text{or } p|b}} \prod_{0 \leq i \leq n} |L_{i,p}(\mathbf{X}_k)|_p < \left| \frac{2(M+1)}{R_k} \right| \leq b^{-\frac{1}{2}\epsilon \lfloor D_k \rfloor} < \|\mathbf{X}_k\|^{-\frac{\epsilon}{4L}}$$

for k sufficiently large. Now by Theorem 3.1, there exist a nonzero element $(e_0, e_1, \dots, e_n) \in \mathbb{Z}^{n+1}$, and an infinite subset \mathbb{N}' of \mathbb{N} such that

$$(1) \quad e_0 Q_k + \sum_{i=1}^n e_i b^{d_{i,1,k}} = 0,$$

for each $k \in \mathbb{N}'$.

As $\langle \overline{D_k} \rangle$ tends to infinity, we can choose an infinite subset \mathbb{N}'' of \mathbb{N}' , and a permutation $\{s_1, \dots, s_n\}$ of $\{1, \dots, n\}$, satisfying:

(i) for each $k \in \mathbb{N}''$,

$$d_{s_1,1,k} > \dots > d_{s_n,1,k};$$

(ii) for each $i < n$,

$$\lim_{\substack{k \in \mathbb{N}'' \\ k \rightarrow \infty}} d_{s_i,1,k} - d_{s_{i+1},1,k} = +\infty.$$

Now dividing Formula (1) by $b^{d_{s_1,1,k}}$ and letting k tend to infinity along \mathbb{N}'' , we obtain

$$\alpha_{s_1} e_0 + e_{s_1} = 0.$$

Hence, by the irrationality of α_1 ,

$$e_0 = e_{s_1} = 0.$$

Finally, dividing Formula (1) by $b^{d_{s_2,1,k}}, \dots, b^{d_{s_n,1,k}}$ in turn, we get

$$e_{s_2} = \dots = e_{s_n} = 0.$$

This concludes the proof of the theorem. \square

Remark 3.2. *The above proof uses only the irrationality of α_{s_1} . Thus if we require all the L -admissible $\underbrace{(1, \dots, 1)}_n$ -arrays $\{d_{i,1}\}$ to satisfy*

$$d_{1,1} > d_{2,1} > \dots > d_{n,1},$$

then Theorem 2.1 still holds if we only assume the irrationality of α_1 .

Proof of Theorem 2.2. Assume that there exists an infinite sequence

$$\{D_k\}_{k \geq 1} = \{\{d_{i,j,k}\}_{i,j}\}_{k \geq 1}$$

of L -admissible (a_1, a_2, \dots, a_n) -arrays, such that both $\lfloor D_k \rfloor$ and $\langle D_k \rangle$ tend to infinity, and

$$(2) \quad R_k \geq b^{\epsilon \lfloor D_k \rfloor},$$

where we set

$$R_k = G.C.D.([f(D_k) + P(D_k)], b^{\lfloor D_k \rfloor}).$$

We can choose an infinite subset \mathbb{N}' of \mathbb{N} , a partition of the set

$$\Phi = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq a_i\} = \bigcup_{s=1}^t U_s,$$

and a fixed $(i_s, j_s) \in U_s$ for each s , such that for each $k \in \mathbb{N}'$,

- (i) $\min_{(i,j) \in U_s} (d_{i,j,k}) = d_{i_s, j_s, k}$, $0 < s \leq t$;
- (ii) $d_{i,j,k} - d_{i_s, j_s, k}$ is independent of k , when $0 < s \leq t$ and $(i, j) \in U_s$;
- (iii) the sequence $\{|d_{i,j,k} - d_{i_s, j_s, k}|\}_{k \in \mathbb{N}'}$ tends to infinity, when $0 < s \leq t$ and $(i, j) \notin U_s$.

As $\langle D_k \rangle$ tend to infinity, we see that $(i, j), (i', j') \in U_s$ and $(i, j) \neq (i', j')$ imply $i \neq i'$. Hence by the assumption on $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$\beta_s = \sum_{(i,j) \in U_s} \alpha_i c_{i,j} b^{d_{i,j,k} - d_{i_s, j_s, k}},$$

is an irrational algebraic number when $k \in \mathbb{N}'$. For $k \in \mathbb{N}'$ we have

$$f(D_k) = \sum_{s=1}^t \beta_s d_{i_s, j_s, k}.$$

Set $E_k = \{d_{i_s, j_s, k}\}_s$. Then $\lfloor E_k \rfloor = \lfloor D_k \rfloor$ when $k \in \mathbb{N}'$. Hence both $\lfloor E_k \rfloor$ and $\langle \overline{E_k} \rangle$ tend to infinity along \mathbb{N}' . Now applying Theorems 2.1 to β_1, \dots, β_t and $\{E_k\}_{k \in \mathbb{N}'}$ implies

$$R_k < b^{\epsilon \lfloor E_k \rfloor} = b^{\epsilon \lfloor D_k \rfloor},$$

when $k \in \mathbb{N}'$ is sufficiently large. This contradicts inequality (2). \square

Proofs of Theorems 2.3 and 2.4. Theorem 2.3 follows by applying Theorem 2.2 to the case $n = 1$ and

$$P(D) = P(\{d_j\}) = - \sum_{j=1}^{a_1} (c_j \{\alpha b^{d_j}\} + \{c_j [\alpha b^{d_j}]\}).$$

Theorem 2.4 follows by applying Theorems 2.2 to the case

$$a_1 = \dots = a_n = 1$$

and

$$P(D) = P(\{d_i\}) = - \sum_{i=1}^n \{\alpha_i b^{d_i}\}.$$

□

Proofs of Theorems 2.7 and 2.10. We prove Theorem 2.7 first. Assume that the b -ary expansion of α satisfies Condition 2.5. Set $d_{i,j} = |A_i^j B_i^j|$. Then each $D_i = \{d_{i,j}\}_j$ is an $(L+1)$ -admissible (m) -array by (v) of Condition 2.5. It follows from (ii) and (iii) of Condition 2.5 that both the sequences $\{\lfloor D_i \rfloor\}_{i \geq 1}$ and $\{\langle D_i \rangle\}_{i \geq 1}$ are strictly increasing. (iv) of Condition 2.5 implies

$$f(D_i) = \sum_{j=1}^m a_j [\alpha b^{d_{i,j}}]$$

is divisible by b^{k_i} , hence

$$G.C.D(f(D_i), b^{\lfloor D_i \rfloor}) \geq b^{k_i} \geq b^{\frac{1}{L+1} \lfloor D_i \rfloor}$$

for each $i \geq 1$. On the other hand, applying Theorem 2.3 to $f(D_i)$ and the sequence $\{D_i\}_{i \geq 1}$ implies

$$G.C.D(f(D_i), b^{\lfloor D_i \rfloor}) < b^{\frac{1}{L+1} \lfloor D_i \rfloor},$$

when i is sufficiently large. This concludes the proof of Theorem 2.7. Theorem 2.10 follows from Theorem 2.4 in exactly the same way. □

Proof of Theorem 1.4. Assume that the b -ary expansions of α and α' satisfy Condition 1.3. It follows from Theorem 2.10 that $1, \alpha$ and α' are not linearly independent over \mathbb{Q} . Thus there exists a nonzero element (x, y, z) in \mathbb{Z}^3 , such that

$$(3) \quad x + y\alpha + z\alpha' = 0.$$

The above equality implies $y, z \neq 0$. Set $|A_n B_n| = l_n$, $|A'_n B_n| = l'_n$, and $|B_n| = k_n$. Without loss of generality, we can choose an infinite subset \mathbb{N}' of \mathbb{N} such that either $l_n - l'_n$ is a fixed integer s , or $\{l_n - l'_n\}$ tends to $+\infty$ along \mathbb{N}' . In the second case, applying Theorem 2.1 to the sum $[\alpha b^{l_n}] - [\alpha' b^{l'_n}]$ implies

$$G.C.D([\alpha b^{l_n}] - [\alpha' b^{l'_n}], b^{l'_n}) < b^{\frac{1}{L+1} l'_n} \leq b^{k_n},$$

when $n \in \mathbb{N}'$ is sufficiently large; this contradicts (i) of Condition 1.3. Hence we assume that $l_n - l'_n$ is a fixed positive integer s for $n \in \mathbb{N}'$. We have

$$[\alpha b^{l_n}] - [\alpha' b^{l'_n}] = \left(\frac{z\alpha b^s + y\alpha + x}{z} \right) b^{l'_n} - (\{\alpha b^{l_n}\} - \{\alpha' b^{l'_n}\})$$

when $n \in \mathbb{N}'$. If

$$z\alpha b^s + y\alpha \neq 0,$$

we can apply Theorem 2.1 to get a contradiction as above. Hence

$$(4) \quad zb^s + y = 0.$$

Now (i) of Condition 1.3 implies that

$$[\alpha b^{l_n}] - [\alpha' b^{l'_n}] = \left[\frac{x}{z} b^{l'_n} \right] - \theta_n$$

4. A GENERALIZED BOREL CONJECTURE AND SOME OTHER QUESTIONS

(i) Theorem 2.10 implies that b -ary expansions of linearly independent irrational algebraic numbers are quite independent. Motivated by this result, we propose a generalized Borel conjecture.

$$\begin{aligned}\alpha_1 &= [\alpha_1] + 0.a_{1,1}a_{1,2}a_{1,3}\cdots \\ \alpha_2 &= [\alpha_2] + 0.a_{2,1}a_{2,2}a_{2,3}\cdots \\ &\vdots \\ \alpha_n &= [\alpha_n] + 0.a_{n,1}a_{n,2}a_{n,3}\cdots\end{aligned}$$
$$A_b(D, N, \xi) := \text{Card}\{i | 1 \leq i \leq N, D_i = D\},$$
$$D_i = \begin{pmatrix} a_{1,i} & a_{1,i+1} & \cdots & a_{1,i+m-1} \\ a_{2,i} & a_{2,i+1} & \cdots & a_{2,i+m-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n,i} & a_{n,i+1} & \cdots & a_{n,i+m-1} \end{pmatrix}.$$
$$\lim_{N \rightarrow +\infty} \frac{A_b(D, N, \xi)}{N} = \frac{1}{b^{mn}},$$

The proof of Borel's theorem by the law of large numbers in [5, p.110] can be used directly to show that:

Theorem 4.1. *Almost all n -tuples of real numbers are normal to base b .*

Motivated by this result and Theorem 2.10, we propose the following generalization of Borel conjecture.

Conjecture 4.2 (Generalized Borel Conjecture).

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n algebraic numbers, such that $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly independent over \mathbb{Q} . Then $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is normal to base b .

It is easy to check that the above conjecture is invalid if $1, \alpha_1, \alpha_2, \dots, \alpha_n$ are linearly dependent over \mathbb{Q} .

- (ii) The proof of Theorem 2.1 and Remark 3.2 implies the following theorem:

Theorem 4.3. *Let k be a positive integer, let ϵ be a positive number, and let*

$$f(x) = a_k x^k + \dots + a_1 x + a_0$$

be a polynomial with real algebraic coefficients, where a_k is irrational. Then we have

$$G.C.D([f(b^n)], b^n) < b^{\epsilon n},$$

when $n \in \mathbb{N}$ is sufficiently large.

It is reasonable to expect that the following more general result also holds:

Conjecture 4.4. *Let ϵ be a positive number, let k, l be two positive integers, and let*

$$f(x) = a_k x^k + \dots + a_1 x + a_0,$$

and

$$g(x) = c_l x^l + \dots + c_1 x + c_0,$$

be polynomials with real algebraic coefficients, where a_k and c_l are linearly independent over \mathbb{Q} . Then we have

$$G.C.D([f(b^n)], [g(b^n)]) < b^{\epsilon n},$$

when $n \in \mathbb{N}$ is sufficiently large.

- (iii) All the congruences in Conditions 2.5, 2.6 and 2.12 are linear. We ask that whether similar results held for some nonlinear congruences. According to Theorems 2.1-2.4, this amounts to similar estimation of the upper bound of greatest common divisor of more general power sums. It is too difficult to give a general formulation of such question. Following are three simplest illustrations.

Conjecture 4.5. *Let α, β be two irrational algebraic numbers, and let ϵ be a positive number. For nonnegative integers k and m , set*

$$f(k, m) = [\alpha b^{k+m}][\beta b^m] + 1.$$

Then we have

$$G.C.D(f(k, m), b^m) < b^{\epsilon m},$$

when m is sufficiently large.

Conjecture 4.6. *Let α , β and ϵ be as before. For two nonnegative integers k and m , set*

$$f(k, m) = [\alpha b^{k+m}]^2 + [\beta b^m].$$

Then we have

$$G.C.D(f(k, m), b^m) < b^{\epsilon m},$$

when m is sufficiently large.

Conjecture 4.7. *Let α , β and ϵ be as before. For positive integer m , set*

$$f(m) = [\beta[\alpha b^m]^2].$$

Then we have

$$G.C.D(f(m), b^m) < b^{\epsilon m},$$

when m is sufficiently large.

Conjectures 4.5 and 4.6, if valid, would provide new evidences for the Generalized Borel Conjecture.

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